Bàn về hệ thống tuyến tính có cấu trúc và bài toán loại bỏ nhiễu

Discussion of linear structured system and the disturbance rejection problem

Đỗ Trọng Hiếu
Trường ĐHBK Hà Nội
e-Mail: hieu.dotrong@hust.edu.vn

Tóm tắt

Bài báo này đề cập đến mô hình hệ thống tuyến tính có cấu trúc và ứng dụng trong nghiên cứu loại bỏ nhiễu bằng phản hồi trạng thái hoặc bằng phản hồi tín hiệu đo. Các điều kiện cần và đủ cho việc kiểm tra khả năng loại bỏ nhiễu bằng phản hồi trạng thái hoặc bằng phản hồi tín hiệu đo được đề xuất. Khi bài toán loại bỏ nhiễu bằng phản hồi đầu đo không khả thi, chúng tôi nghiên cứu bài toán xác định số lượng cảm biến cần thêm vào và trạng thái cần đo để bài toán loại bỏ nhiễu bằng phản hồi đầu đo trở nên khả thi. Các phân tích và kết quả trong bài báo được thể hiện trong khuôn khổ hệ thống tuyến tính có cấu trúc.

Từ khóa: Hệ thống tuyến tính có cấu trúc, đặc tính chung, xác định vị trí cảm biến, loại bỏ nhiễu.

Abstract: In this paper we present the structured linear system and revisit the exact disturbance rejection problem in a structural framework. Necessary and sufficient conditions are proposed for the solvability of the Disturbance Rejection by State Feedback (DRSF) problem and the Disturbance Rejection by Measurement Feedback (DRMF) problem. The associated system graph can be used to easily check whether or not the conditions hold. When the DRMF problem is not solvable, we investigate how many sensors are needed and where should they be located to make this problem solvable. Our analysis is performed in the context of structured systems which represent a large class of parameter dependent linear systems. This structured system gives us more understanding of the system.

Keywords: Linear structured systems, structural properties, sensor location, disturbance rejection.

Chữ viết tắt

DRSF	disturbance rejection by state feedback
DRMF	disturbance rejection by measurement feedback

1. Introduction

We consider here linear structured systems which represent a large class of parameter dependent linear systems. Generic properties for such systems can be obtained easily from a graph naturally associated with the systems. This approach was pioneered by Lin [6]. In this framework, the DRMF problem has been solved via a graph approach in [7], [8]. It is clear that the solvability of this problem highly relies on the sensor network. Sensor location has already been studied in a structural framework for two other problems, the observability in [9], [10], [11] and the Fault Detection and Isolation problem in [12], [13].

Dynamic systems are often affected by unmeasurable disturbances. It is important that some system performances are still performed in the presence of disturbances. Control of physical systems must take into account the existence of disturbances and possibly reject their effect. This paper is concerned with a classical problem of the control theory of linear systems, called the exact disturbance rejection problem (*i.e.* a zero disturbance-regulated output transfer matrix). To eliminate the influence of disturbances on the regulated output of the system, it is necessary to have information on disturbances and their effect on system. Normally, this information is obtained from measurements (using sensors). In the case where all states are measurable, we have the problem of disturbance rejection by state feedback. Otherwise, there is the problem of disturbance rejection by measurement feedback. Other approaches allow to stabilize and minimize some norm of disturbance-regulated output transfer matrix, see for example [1]. The problem of disturbance rejection by state feedback is a very well known problem [2], [3]. In the case where the state is not available for measurement, the problem is more complex. The problem of disturbance rejection by measurement feedback has been solved in an elegant way in geometric terms, see [4], [5].

In this paper we present the structured linear system and revisit the disturbance rejection problem (by state feedback and by measurement feedback) in the context of linear system and then of linear structured system. Necessary and sufficient conditions for the problem has a solution are presented. In the DRMF problem, we prove that the problem reduces to an unknown input observer problem on a subset of the state space. This subset consists of the states for which a disturbance affecting directly theses states can be rejected by state feedback. The observation problem amounts to estimate the disturbance effect before it leaves this subset. This allows to explain why we need to measure a sufficient number of state variables early enough to be able to estimate the disturbance effect and compensate for it via the control input. Consequently, we give the minimal number of sensors to be implemented for solving the DRMF problem. We showed also that the sensors measuring only states out of a given subset are useless for solving the DRMF problem. Our analysis comes within the context of structured systems which represent a large class of linear systems. The generic results are obtained directly from the system associated graph.

The outline of this paper is as follows. We formulate the problem of disturbance rejection in section 2. The linear structured systems are presented in section 3 as well as the known structural results on the DRSF problem and the DRMF problem. The sensor location problem is considered in section 4: we give the minimal number of sensors for solving the DRMF problem and characterize an important set of useless sensors. An illustrative example is given in section 5. Some concluding remarks end the paper.

2. Disturbance Rejection Problem

2.1 Disturbance rejection by state feedback

We consider the linear system Σ given by:

$$\Sigma \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Ed(t) \\ y(t) = Cx(t) \end{cases} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, $d(t) \in \mathbb{R}^q$ is the disturbance, $y(t) \in \mathbb{R}^p$ is the regulated output.

The problem of disturbance rejection by state feedback amounts to find a state feedback of the form $u(t) = Fx(t)$ such that in closed loop the disturbances will have no effect on the regulated output:

$$C(sI - A - BF)^{-1}E = 0 \quad (2)$$

2.2 Disturbance rejection by measurement feedback

When not all the state can be measurable, we have the DRMF problem. Consider the linear system Σ^z given by:

$$\Sigma^z \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Ed(t) \\ y(t) = Cx(t) \\ z(t) = Hx(t) \end{cases} \quad (3)$$

where $u(t) \in \mathbb{R}^m$ is the control input, $d(t) \in \mathbb{R}^q$ is the disturbance, $x(t) \in \mathbb{R}^n$ is the state, $y(t) \in \mathbb{R}^p$ is the regulated output and $z(t) \in \mathbb{R}^n$ is the measured output provided by a sensor network.

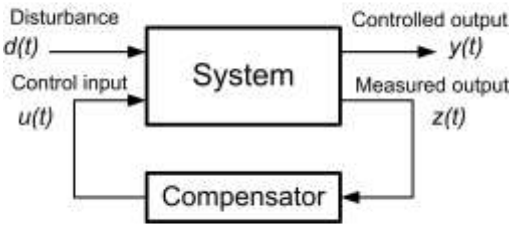
For such a system, we have the transfer matrix:

$$\begin{bmatrix} y(s) \\ z(s) \end{bmatrix} = \begin{bmatrix} G(s) & K(s) \\ M(s) & N(s) \end{bmatrix} \begin{bmatrix} u(s) \\ d(s) \end{bmatrix} \quad (4)$$

The problem of disturbance rejection amounts to find a dynamic measured output feedback compensator

$$\Sigma_{zu} \begin{cases} \dot{w}(t) = Lw(t) + Rz(t) \\ u(t) = Sw(t) + Pz(t) \end{cases} \quad (5)$$

such that in closed loop the disturbances will have no effect on the regulated output.



H. 1 *Control by dynamic feedback compensation*

In transfer matrix terms, we look for a dynamic compensator (see H. 1) u(s) = F(s)z(s), where F(s) is a proper rational matrix, such that the closed loop system transfer matrix from disturbance d to controlled output y is identically zero:

$$G(s)F(s)(I - M(s)F(s))^{-1}N(s) + K(s) = 0 \quad (6)$$

This problem received a very elegant solution in geometric terms, see [4]. A geometric necessary and sufficient condition for the solvability of the disturbance rejection by measurement feedback problem is:

$$\eta^* \subset \vartheta^* \quad (7)$$

where η^* is the minimal (H,A)-invariant subspace containing ImE and ϑ^* is the maximal (A,B)-invariant subspace contained in KerC.

In the following, we revisit the disturbance rejection problem in a structural way. In the case of DRMF, we give some understandings and useful information on the minimal number of sensors to be implemented and on their possible location.

3. Linear structured system

3.1 Definitions

In this subsection we recall some definitions and results on linear structured systems. More details can be found in [14].

We consider linear systems of type (1) with parameterized entries and denoted by Σ_Λ as follows:

$$\Sigma_\Lambda \begin{cases} \dot{x}(t) = A_\Lambda x(t) + B_\Lambda u(t) + E_\Lambda d(t) \\ y(t) = C_\Lambda x(t) \end{cases} \quad (8)$$

This system is called a linear structured system if the entries of the composite matrix $J_\Lambda = \begin{bmatrix} A_\Lambda & B_\Lambda & E_\Lambda \\ C_\Lambda & 0 & 0 \end{bmatrix}$ are either fixed zeros or independent parameters (not related by algebraic equations). $\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_k\}$ denotes the set of independent parameters of the composite matrix J_Λ.

For such systems, one can study *generic properties*, *i.e.* properties which are true for almost all values of the parameters collected in Λ [15]. More precisely, a property is said to be generic (or structural) if it is true for all values of the parameters outside a proper algebraic variety of the parameter space.

A directed graph $G(\Sigma_\Lambda) = (V, W)$ can be associated with the structured system of type (8):

- the vertex set is $V = U \cup D \cup X \cup Y$ where U, D, X, and Y are the input, disturbance, state and regulated output sets given by $\{u_1, u_2, ..., u_m\}$, $\{d_1, d_2, ..., d_q\}$, $\{x_1, x_2, ..., x_n\}$ and $\{y_1, y_2, ..., y_p\}$ respectively,
- the arc set is W= $\{(u_i, x_j) | B_{\Lambda,ji} \neq 0\}$ U $\{(d_i, x_j) | E_{\Lambda,ji} \neq 0\}$ U $\{(x_i, x_j) | A_{\Lambda,ji} \neq 0\}$U $\{(x_i, y_j) | C_{\Lambda,ji} \neq 0\}$ where $A_{\Lambda,ji}$ (resp. $B_{\Lambda,ji}, E_{\Lambda,ji}, C_{\Lambda,ji}$) denotes the entry (j,i) of the matrix A_Λ (resp. B_Λ, E_Λ, C_Λ)

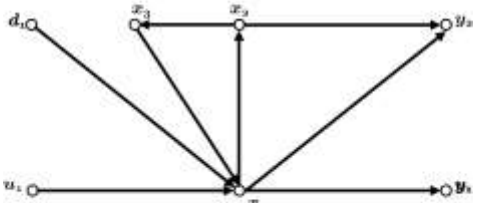
Let V_1, V_2 be two nonempty subsets of the vertex set V of the graph $G(\Sigma_\Lambda)$.We say that there exists a *path* from V_1 to V_2 if there are vertices $i_0, i_1, ..., i_r$ such that $i_0 \in V_1, i_r \in V_2, i_t \in V$ for $t = 0,1,...,r$ and $(i_{t-1}, i_t) \in W$ for $t = 1,2,...,r$. We call the path *simple* if every vertex on the path occurs only once. Two paths from V_1 to V_2 are said to be *disjoint* if they consist of disjoint sets of vertices. r paths from V_1 to V_2 are said to be disjoint if they are mutually disjoint, *i.e.* any two of them are disjoint. A set of r disjoint and simple paths from V_1 to V_2 is called a *linking from* V_1 *to* V_2 *of size* r.

Example 1: Consider the following example of a structured system whose matrices of Equation (8) are the following:

$$A_\Lambda = \begin{bmatrix} 0 & 0 & \lambda_1 \\ \lambda_2 & 0 & 0 \\ 0 & \lambda_3 & 0 \end{bmatrix}, B_\Lambda = \begin{bmatrix} \lambda_4 \\ 0 \\ 0 \end{bmatrix}, E_\Lambda = \begin{bmatrix} \lambda_8 \\ 0 \\ 0 \end{bmatrix}$$

$$C_\Lambda = \begin{bmatrix} \lambda_5 & 0 & 0 \\ \lambda_6 & \lambda_7 & 0 \end{bmatrix}$$

The associated graph $G(\Sigma_\Lambda)$ is depicted in H. 2. In this example, there is only one D − Y path: (d_1, x_1, y_1) then the maximal size of a D − Y linking is one.



H. 2 *Directed graph* $G(\Sigma_\Lambda)$ *of Example 1*

3.2 Disturbance rejection by state feedback for structured system

In order to solve the disturbance rejection problem in the context of structured systems, we will define the first important sets of vertices in the graph $G(\Sigma_\Lambda)$.

Definition 1: Consider Σ_Λ a structured system of type (8) with associated graph $G(\Sigma_\Lambda)$. Let us define the vertex set I^* as follows:

$I^* = \{ x_i \in X$ | the maximal size of a linking in $G(\Sigma_\Lambda)$ from $U \cup \{x_i\}$ to Y is the same as the maximal size of a linking in $G(\Sigma_\Lambda)$ from U to Y , and the minimal number of vertices in $X \cup U$ is the same for both such maximal linkings$\}$.

The set I^* corresponds to the states for which an unmeasurable disturbance affecting directly these states can be rejected by state feedback [8]. Notice that I^* can be computed independently of the sensor network since its computation involves uniquely the matrices A_Λ, B_Λ and C_Λ in (8).

With the definition of I^*, the solubility of the DRSF problem was graphically characterized in [8]:

Theorem 1: Consider Σ_Λ a structured system of type (8) with associated graph $G(\Sigma_\Lambda)$. The problem of disturbance rejection by state feedback is generically solvable if and only if the disturbances affect only state vertices of I^*, *i.e.* for any $(d_i, x_j) \in W, x_j \in I^*$.

From the definitions of I^*, checking condition in Theorem 1 amounts to compute in $G(\Sigma_\Lambda)$ some maximal linkings with minimal number of vertices. This can be done using standard algorithms of combinatorial optimization as max-flow min-cost techniques [16], [17]. This means that for a given sensor network, the solvability of the DRSF problem can be checked in polynomial time.

Return to the structured system in Example 1 with the directed graph $G(\Sigma_\Lambda)$ presented in H. 2. The vertex set I^* can be calculated due to Definition 1. The maximal size of a linking from U to Y is 1 and the minimal number of vertices in $X \cup U$ in such a linking is 2, for example the path (linking of size 1) (u_1, x_1, y_1).

A maximal linking from $U \cup \{x_1\}$ to Y is of size 1 but the minimal number of vertices in $X \cup U$ in such a linking is 1, for example the linking (x_1, y_1). Then $x_1 \notin I^*$.

A maximal linking from $U \cup \{x_2\}$ to Y is of size 2 with the linking (u_1, x_1, y_1), (x_2, y_2). Then $x_2 \notin I^*$.

The state vertex $x_3 \in I^*$ since a maximal linking from $U \cup \{x_3\}$ to Y is of size 1 and the minimal number of vertices in $X \cup U$ in such a linking is 2.

We obtain $I^* = \{x_3\}$. The disturbance d_1 arrives on state vertex $x_1 \notin I^*$ then by Theorem 1, the problem of disturbance rejection by state feedback is generically not soluble.

3.3 Disturbance rejection by measurement feedback for structured system

The linear system of the form (3) can be redefined in structured context. Consider linear systems of type (3) with parameterized entries and denoted by Σ_Λ^z as follows:

$$\Sigma_\Lambda^z \begin{cases} \dot{x}(t) = A_\Lambda x(t) + B_\Lambda u(t) + E_\Lambda d(t) \\ y(t) = C_\Lambda x(t) \\ z(t) = H_\Lambda x(t) \end{cases} \tag{9}$$

This system is called a linear structured system if the entries of the composite matrix $J_\Lambda^z = \begin{bmatrix} A_\Lambda & B_\Lambda & E_\Lambda \\ C_\Lambda & 0 & 0 \\ H_\Lambda & 0 & 0 \end{bmatrix}$ are either fixed zeros or independent parameters (not related by algebraic equations). $\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_h\}$ denotes the set of independent parameters of the composite matrix J_Λ.

For this linear structured system, we can associate a directed graph $G(\Sigma_\Lambda^z) = (V', W')$ in the same manner as the directed graph $G(\Sigma_\Lambda) = (V, W)$ in section 3.1, *i.e.*

- the vertex set $V' = V \cup Z$ where Z is the measured output set given by $\{z_1, z_2, ..., z_n\}$,
- the arc set is $W' = W \cup W_{XZ}$ where $W_{XZ} = \{(x_i, z_j) | H_{\Lambda,ji} \neq 0\}$, $H_{\Lambda,ji}$ denotes the entry (j,i) of the matrix H_Λ.

Note that the determination of I^* from Definition 1 depends only on the matrices A_Λ, B_Λ and C_Λ. Therefore, the vertex set I^* in $G(\Sigma_\Lambda^z)$ is the same as in $G(\Sigma_\Lambda)$. Recall that I^* characterizes the states for which an unmeasurable disturbance affecting directly these states can be rejected by state feedback.

A condition for the DRMF problem is derived in [8]. However, this condition does not provide much information on the solvability of the DRMF problem with respect to the possible location of sensors. It means, when the problem of DRMF is not soluble, where and how many new sensors can be added such that this problem becomes soluble. Therefore, in [18] we revisited this problem and gave alternative necessary and sufficient solvability condition. This condition will give new insight into the problem and provide with useful information on the number and the location of the sensors to be implemented. Let us give first some definitions.

Definition 2: Consider Σ_Λ^z a structured system of type (9) with associated graph $G(\Sigma_\Lambda^z)$. Define F_{I^*}, the frontier of I^*, as the set of vertices

$F_{I^*} = \{x_i \in I^* | \exists (x_i, x_j) \in W', x_j \notin I^*\}$

The set F_{I^*} contains the vertices of I^* which have at least one successor outside of I^*. Practically, the frontier F_{I^*} connects the vertices of I^* with the state vertices which are outside of I^*. If the disturbance affects a vertex in I^*, the effect of the disturbance will propagate firstly in I^* and then must go through F_{I^*} before going out of I^*.

Definition 3: Consider Σ_Λ^z a structured system of type (9) with associated graph $G(\Sigma_\Lambda^z)$. For a disturbance d_i which affects at least one vertex in I^*, denote r_i (*resp.* $l(d_i, x_j)$) the length of a shortest path from d_i to F_{I^*} (*resp.* from d_i to $x_j \in I^*$) where the length of a path is the number of arcs it is composed of. Define D_i the set of vertices:

$D_i = \{x_j \in I^* | 0 < l(d_i, x_j) \leq r_i\}$

We call this set the *disc* associated with the disturbance d_i.

In fact, when a disturbance d_i affects a vertex in I^*, its effect will propagate outside I^*. The set D_i defined above contains the states that have been affected by the disturbance d_i when the effect of this disturbance reaches the frontier F_{I^*}

In the following theorem we give a new insight into the DRMF problem and prove that it is sufficient to study this problem on a part of the state space [18].

Theorem 2: Consider Σ_Λ^z a structured system of type (9) with associated graph $G(\Sigma_\Lambda^z)$ and affected by the disturbances $d_1, ..., d_q$. The DRMF problem is generically solvable if and only if:

- $d_1, ..., d_q$, affect only state vertices of I^*, *i.e.* for any $(d_i, x_j) \in W'$, $x_j \in I^*$.
- The maximal size of a linking in $G(\Sigma_\Lambda^z)$ from D to Z is the same as the maximal size of a linking in $G(\Sigma_\Lambda^z)$ from D to $Z \cup F_{I^*}$, and the minimal number of vertices in X is the same for both such maximal linkings.

Interpretation:

Since the first condition is a necessary and sufficient condition for the solvability of the DRSF [8], it is necessary also for DRMF problem. The second condition corresponds, in graphic terms and within our framework, to an Unknown Input Observer problem [3], [19]. The condition expresses the fact that it is possible to estimate the effect of the disturbances at F_{I^*} from the measurements without the knowledge of the disturbances. It is a natural condition since when the effect of the disturbances at F_{I^*} cannot be estimated from the available measurements, the unknown effect of a disturbance on F_{I^*} will propagate out of I^* and cannot be rejected by state feedback and consequently by measurement feedback. The first part of the condition is a rank condition. When it is not satisfied, we therefore need more sensors to solve the problem. The second part of the condition, when not satisfied, means that the sensors give information on the disturbance too late.

4. Sensor location for the disturbance rejection by measurement feedback

In this section we will examine the consequences of Theorem 2 on the possible sensor location for solving the disturbance rejection by measurement feedback problem. The first result shows that it is useless to measure variables outside I^*.

Proposition 1: Consider Σ_Λ^z a structured system of type (9) with associated graph $G(\Sigma_\Lambda^z)$. Assume that the DRMF problem is generically solvable. A sensor $z_j \in Z$ such that for any $(x_i, z_j) \in W', x_i \in X \setminus I^*$ (*i.e.* z_j measures only states out of I^*) is of no use for solving the DRMF problem.

State now a result giving the minimal number of sensors to be implemented [18].

Proposition 2: Consider Σ_Λ^z a structured system of type (9) with associated graph $G(\Sigma_\Lambda^z)$. The problem of DRMF is generically solvable only if the number of sensors is greater than or equal to the maximal size of a linking from D to F_{I^*} in $G(\Sigma_\Lambda^z)$.

The following result shows that it is necessary to have a measurement in each disc D_i associated with disturbance d_i. Remark that one measurement can be valid for several discs.

Proposition 3: Consider Σ_Λ^z a structured system of type (9) with associated graph $G(\Sigma_\Lambda^z)$. The problem of DRMF is generically solvable only if for any D_i, $i = 1, 2, ..., q$, there exists $x^* \in D_i$ and $z^* \in Z$ such that the arc $(x^*, z^*) \in W'$.

The following theorem proves that measuring states of I^* sufficiently close to the disturbances and in a decoupled manner is sufficient to solve the DRMF problem.

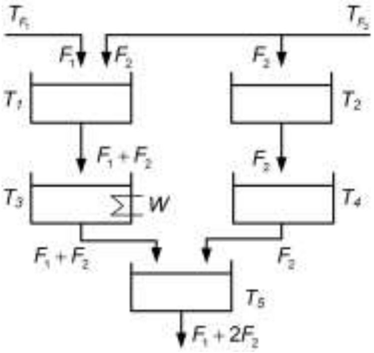
Proposition 4: Consider Σ_Λ^z a structured system of type (9) with associated graph $G(\Sigma_\Lambda^z)$ and affected by the disturbances $d_1, ..., d_q$. Assume that the disturbances affect only state vertices of I^*, *i.e.* for any $(d_i, x_j) \in W', x_j \in I^*$. Then the DRMF problem is generically solvable if in $G(\Sigma_\Lambda^z)$ there exists a linking of size q from D to q vertices of Z *i.e.* $(d_1, ..., z_{i1})$, $(d_2, ..., z_{i2})$, …, $(d_q, ..., z_{iq})$ such that:

- $(d_j, ..., z_{ij})$ is a shortest path from d_j $(j = 1, ..., q)$ to Z.
- the number of state vertices in the path $(d_j, ..., z_{i_j})$ is lower than or equal to r_j, the number of state vertices in a shortest path from d_j to F_{I^*}.

5. Disturbance rejection for a thermal process

5.1 The system

Consider the thermal process described in H.3.



H. 3 *The system made up of 5 tanks*

This process consists of five tanks such that each tank is fed by a fixed water flow: $(F_1 + F_2)$ for tank 1 and 3, F_2 for tanks 2 and 4, $(F_1 + 2F_2)$ for tank 5. The system control input is the heating power W. The regulated output is T_5, the temperature of the fifth tank. The disturbances are the variations of feed flow temperatures T_{F_1} and T_{F_2}. The objective is to determine a dynamic measured output feedback such that T_5 is not sensitive to the variations of T_{F_1} and T_{F_2}.

This process can be linearized around a given operating point as a system of the type defined by (1) where

$$x = [\Delta T_1 \quad \Delta T_2 \quad \Delta T_3 \quad \Delta T_4 \quad \Delta T_5]^T, u = \Delta W$$

$$d = [\Delta T_{F_1} \quad \Delta T_{F_2}]^T, y = \Delta T_5$$

and the state matrices

$$A=\begin{bmatrix} -\frac{(F_1+F_2)}{C_1} & 0 & 0 & 0 & 0 \\ 0 & \frac{-F_2}{C_2} & 0 & 0 & 0 \\ \frac{(F_1+F_2)}{C_3} & 0 & \frac{-(F_1+F_2)}{C_3} & 0 & 0 \\ 0 & \frac{F_2}{C_4} & 0 & \frac{-F_2}{C_4} & 0 \\ 0 & 0 & \frac{(F_1+F_2)}{C_5} & \frac{F_2}{C_5} & \frac{-(F_1+2F_2)}{C_5} \end{bmatrix}$$

$$B=\begin{bmatrix} 0 \\ 0 \\ 1/C_3 \\ 0 \\ 0 \end{bmatrix}, E=\begin{bmatrix} F_1/C_1 & F_2/C_1 \\ 0 & F_2/C_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C=\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

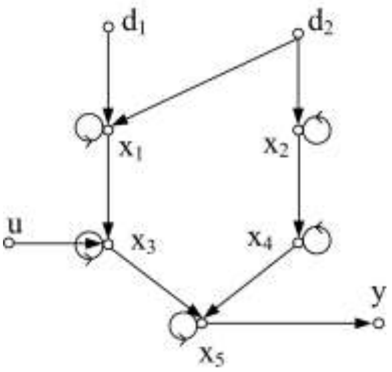
C_i is the heat capacity of the i^{th} tank.

This model clearly exhibits the physical structure of the process. Note that this model is not exactly structured as in Section 3 since some dependencies exist between the matrix entries. Nevertheless, in order to illustrate the approach, we will consider a structured system of the form defined by equations (8) that has the same zero/nonzero structure as the physical system with the following matrices:

$$A=\begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ \lambda_3 & 0 & \lambda_4 & 0 & 0 \\ 0 & \lambda_5 & 0 & \lambda_6 & 0 \\ 0 & 0 & \lambda_7 & \lambda_8 & \lambda_9 \end{bmatrix}, B=\begin{bmatrix} 0 \\ 0 \\ \lambda_{10} \\ 0 \\ 0 \end{bmatrix}, E=\begin{bmatrix} \lambda_{11} & \lambda_{12} \\ 0 & \lambda_{13} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C=\begin{bmatrix} 0 & 0 & 0 & 0 & \lambda_{14} \end{bmatrix}$$

The associated graph of this structured system is depicted in H. 4. Note that this system is not controllable since the state vertices x_1, x_2 and x_4 are not relied to the input vertex u.



H. 4 *The system made up of 5 tanks*

5.2 The solvability of the disturbance rejection

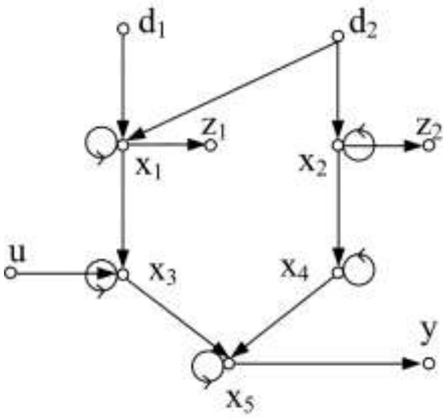
From Definition 1, we obtain $I^* = \{x_1, x_2\}$. We can verify on the associated graph in H. 4 that the disturbances affect directly only state vertices of I^*. According to Theorem 1, the problem of DRSF generically solvable.

We will now study the sensor location problem for the DRMF. From Definition 2 and Definition 3, we have the frontier $F_{I^*} = \{x_1, x_2\}$, the *disc* associated with the disturbance d_1 is $D_1 = \{x_1\}$ and the *disc* associated with the disturbance d_2 is $D_2 = \{x_1, x_2\}$.

By Proposition 1, the sensors which measure only vertices x_3, x_4 and x_5, *i.e.* which measure only outside I^* are useless for the solubility of the DRMF.

A maximal linking from D to F_{I^*} corresponds to $\{(d_1, x_1), (d_2, x_2)\}$ which is of size 2. From Proposition 2, one needs at least two sensors to reject the disturbance by measurement feedback. Moreover, by Proposition 3 there must be at least one measure in each disc $D_1 = \{x_1\}$ and $D_2 = \{x_1, x_2\}$. With z_1 and z_2 as shown in H. 5, we satisfy the conditions of Proposition 4 and the DRMF problem is solvable. With these measurements, we obtain:

$$H_L = \begin{bmatrix} \lambda_{15} & 0 & 0 & 0 & 0 \\ 0 & \lambda_{16} & 0 & 0 & 0 \end{bmatrix}$$



H. 5 *Graph of the five-tank system with measurements*

Indeed, it turns out that on this model, the measurement of x_1 and x_2 provides early information on the disturbances that allows us to compensate in time with u the effect of these disturbances on the regulated output y.

5.3 Calculation of the dynamic measured output feedback compensator

Here, the matrices in (4) are:

$$G_L(s) = \frac{\lambda_7 \lambda_{10} \lambda_{14}}{(s-\lambda_4)(s-\lambda_9)}, M_L(s) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$K_L(s) = \begin{bmatrix} \dfrac{\lambda_3 \lambda_7 \lambda_{11} \lambda_{14}}{(s-\lambda_1)(s-\lambda_4)(s-\lambda_9)} \\ \dfrac{\lambda_{14}}{(s-\lambda_9)}\left(\dfrac{\lambda_3 \lambda_7 \lambda_{12}}{(s-\lambda_1)(s-\lambda_4)} + \dfrac{\lambda_5 \lambda_8 \lambda_{13}}{(s-\lambda_2)(s-\lambda_6)}\right) \end{bmatrix}^T$$

$$N_L(s) = \begin{bmatrix} \frac{\lambda_{11}\lambda_{15}}{(s-\lambda_1)} & \frac{\lambda_{12}\lambda_{15}}{(s-\lambda_1)} \\ 0 & \frac{\lambda_{13}\lambda_{16}}{(s-\lambda_2)} \end{bmatrix}$$

Therefore in this example, equation (6) can be reduced to:

$$G_L(s)F_L(s)N_L(s) + K_L(s) = 0$$

We then obtain $F_L(s) = \begin{bmatrix} \frac{-\lambda_3}{\lambda_{10}\lambda_{15}} & \frac{-\lambda_5\lambda_8}{\lambda_7\lambda_{10}\lambda_{16}}\left(\frac{s-\lambda_4}{s-\lambda_6}\right) \end{bmatrix}$.

Let $g_1 = \frac{-\lambda_3}{\lambda_{10}\lambda_{15}}$, $g_2 = \frac{-\lambda_5\lambda_8}{\lambda_7\lambda_{10}\lambda_{16}}$, $g_3 = \frac{-\lambda_5\lambda_8(\lambda_6-\lambda_4)}{\lambda_7\lambda_{10}\lambda_{16}}$

then we can get the following realization for the dynamic measured output feedback compensator:

$$\Sigma_{zu} \begin{cases} \dot{w}(t) = \lambda_6 w(t) + \begin{bmatrix} 0 & g_3 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} \\ u(t) = w(t) + \begin{bmatrix} g_1 & g_2 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} \end{cases}$$

The results regarding solvability of the DRMF problem only guarantee that the transfer from the disturbance to the regulated output is null but do not take into consideration the stability issues. That is why this approach needs further analysis to be applied in practice. With the above compensator, we have the dynamic matrix of the closed loop system with extended state $x_e(t) = \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}$

$$A_L^{CL} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ (\lambda_3+\lambda_{10}g_1) & \lambda_{10}g_2 & \lambda_4 & 0 & 0 & \lambda_{10} \\ 0 & \lambda_5 & 0 & \lambda_6 & 0 & 0 \\ 0 & 0 & \lambda_7 & \lambda_8 & \lambda_9 & 0 \\ 0 & g_3 & 0 & 0 & 0 & \lambda_6 \end{bmatrix}$$

The characteristic polynomial of the closed loop system is:

$$\det(sI - A_L^{CL}) = (s-\lambda_1)(s-\lambda_2)(s-\lambda_4)(s-\lambda_9)(s-\lambda_6)^2$$

Since $\lambda_1, \lambda_2, \lambda_4, \lambda_6, \lambda_9$ are negative by nature, *i.e.* for any positive value of the physical parameters, the closed loop system is stable.

5.4 Simulation results

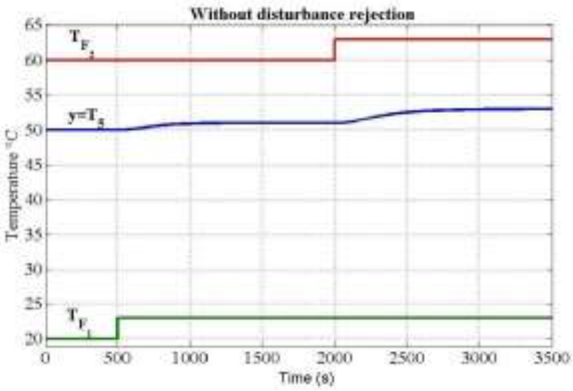
We will now test the system using the following physical values:

- All the tanks volumes are 5l which leads to a heat capacity of $20.93 JK^{-1}$.
- The flow rates are $F_1 = F_2 = 0.1$ l/s
- The feed flow temperatures are $T_{F_1} = 20 \pm 5°C$ and $T_{F_2} = 60 \pm 5°C$.
- Our set point corresponds to a temperature of T_5=50°C with two feed flows at $T_{F_1} = 20°C$, $T_{F_2} = 60°C$, which implies a heating power of u = 4.186kW.

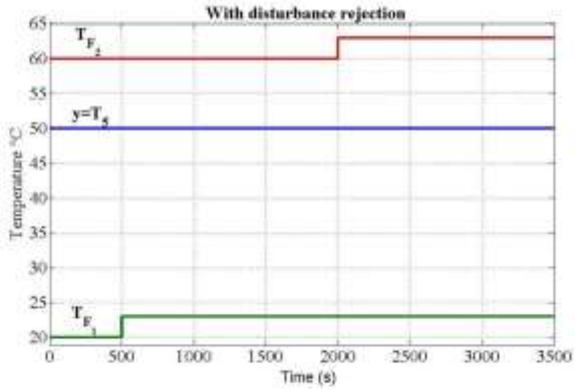
Our aim is to maintain the output temperature at T_5=50°C for any variation of the feed flow temperatures T_{F_1} and T_{F_2}. H.6 shows step disturbances on the temperature feed flows and the open-loop effect on the regulated output. H.7 shows the behaviour of the closed loop system with the DRMF controller.

6. Concluding remarks

In this paper we revisited the disturbance rejection problem in a structural way. We gave some understandings and useful information about this topic. The necessary and sufficient conditions for the problem to be solvable were given. In the DRMF case, we showed that the problem reduces to an unknown input observer problem on a subset of the state space. This structural result allowed us to study the DRMF problem irrespective of the sensors network and then to determine the minimal number of sensors to be implemented and to show that it is useless for the problem to measure states in some region of the state space. Finally, we provided with a constructive sensor network configuration which solves the DRMF problem. This last result is useful in practice for a sensor network design but remains only sufficient.



H. 6 *Temperature of feed flows and regulated output without measurement feedback*



H. 7 *Temperature of feed flows and regulated output with measurement feedback*

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DO Trong Hieu was born in 1984 in Hanoi, Vietnam. He obtained his Electrical Engineering degree from the Polytechnic Institute of Grenoble (Grenoble-INP), France in 2008. From 2008 to 2011 he was a Ph.D. student in the GIPSA-lab of Grenoble, France. The subject of his research was the application of structured systems to sensor location and sensor classification.